**Evidence for Bar Modeling Curriculum Efficacy**

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## 1. Curricular Goals

It is widely understood that learning algebra is essential for future gains in mathematics and pursuit of studies and careers in STEM fields (Lucariello et. al 2014, 30). However, learning algebra is difficult for many students. Common Core Standards (both national and state standards) are including the skills of understanding variables and algebraic expressions as early as 6th grade and solving multi-step equations by 8th grade ("Academic Standards (K-12)" 2007). Students may struggle to solve algebra word problems if they have "difficulty understanding embedded concepts in the problems, have calculation skill deficits, and do not use strategies effectively" (Mornin et. al 2017, 2). Thus, in order to successfully solve algebra problems, students must develop procedural and conceptual knowledge (Booth & Davenport 2013, Hattikudur et. al 2016, Jonsson 2014). The goal of this bar modeling curriculum is to reinforce procedure for solving algebra problems, while developing student's conceptual knowledge of the relationships between variables and numbers in equations as well as understanding of the underlying reasons for the procedure. We also hope to address several misconceptions that students may have including misconceptions of the equal sign, variables, and the negative sign.

This project emerged as a request from a middle school math teacher in Northfield Minnesota to create an algebra curriculum using bar modeling, as adapted from the Singapore method of teaching math. This review will first discuss Singapore math and evidence for its efficacy, and then provide an overview of the relationship and tradeoff between conceptual and procedural knowledge, the gains from learning multiple procedures and finally common algebra misconceptions. This literature review is meant to accompany the full workbook that we created, which can be found at, https://sites.google.com/site/adventuresinalgebra/, so this paper will not include explicit explanation of the curriculum contents.

**2. Singapore Math**

There is evidence that the Singapore method of bar modeling increases math competency and use of cognitive tools to solve algebra problems (Mornin et.al 2017). The evidence suggests that bar modeling can encourage gains in conceptual knowledge and address common algebra misconceptions. Most bar modeling in the US, also sometimes called tape modeling, is based on the Singapore method. In this method, students build a visual model that represents the arithmetic required to solve a word problem. The success of Singapore math not only lies in the visualization method, but in the way they structure the entire curriculum from 5 years old on (Ginsburg 2005). Our curriculum uses a different approach to bar modeling, which more closely mirrors the procedure and equation set up used in the methods to solve algebra problems currently taught in the United States, including explicit definitions of variables in the model. Therefore, our evaluation of the potential efficacy of the approach is based in the proven efficacy of the Singapore method, along with a review of literature on student's difficulties with algebraic problem solving.

**3. Conceptual and Procedural Knowledge**

Conceptual and procedural knowledge are both vital for algebra success. We can define procedural knowledge as knowledge of what steps can be used to solve a problem and when to use them (Booth & Davenport 2013, Hattikudur et. al 2016) while we will define conceptual knowledge as understanding important principles of mathematics (Booth & Davenport), including “principles that underlie procedures” (Hattikudur et. al). Generally procedural and conceptual knowledge “develop in an iterative fashion, with gains from one leading to gains in the other” (Booth & Davenport 2013). Booth & Davenport (2013) showed that in algebra, conceptual knowledge is related to better equation solving abilities. Additionally, being comfortable with a procedure can allow students to apply concepts to solve difficult or novel problems. Thus, while conceptual and procedural knowledge are generally considered to be separate, they are intimately related. It is imperative for students to gain conceptual knowledge to establish strong procedural knowledge and succeed at solving a variety of algebra equations but it is also helpful for students to have established procedural knowledge to free cognitive space to apply other math concepts in solving more challenging problems.

While both forms of knowledge are important, unfortunately in the traditional classroom in the United States, there are time restraints that often require teachers to choose to spend time and energy ensuring that students have procedural knowledge only, rather than procedural and conceptual knowledge. Teaching algorithmic methods to solve problems (targeting procedural knowledge) and repeating this algorithm many times without reflection of the underlying principles that inform the algorithm can lead to students solving these specific problems without any application of conceptual knowledge (Jonsson et. al 2014). While students in the long run will not benefit from only learning procedures, there are cognitive benefits for students to memorize procedures in order to solve problems. Memorizing algorithmic procedures reduces cognitive load, thereby reserving more working memory to apply other knowledge to solving more advanced problems (Jonsson et. al 2014). Additionally, this type of learning is efficient in the short term, which makes it often necessary for teachers to implement, because of the pressure to get students able to pass exams and standardized tests. There is a trade-off between teaching efficient methods that allow students to spend less working memory on determining which procedures to use (algorithmic learning) and encouraging understanding of the principles that underlie procedures (conceptual knowledge).

The strength of our version of bar modeling lies in its ability to enforce procedural knowledge, as the procedures used to manipulate the bars mimic the procedures used to solve a written equation, but add to conceptual knowledge of equation structure. By visualizing the structure of equations with variables and numbers, students should gain a better understanding of equality as a relational symbol and also see that each step in the standard procedure for solving an equation is done to isolate the variable.

**4. Multiple Problem Solving Strategies**

This bar modeling curriculum will also provide students with an alternative strategy to solve problems. There is evidence that learning multiple strategies and procedures to solve problems, especially when strategies are compared, can encourage greater conceptual understanding of material (Hattikudur et. al 2016). Specifically, Hattikudur et. al (2016) found that for students who have negative attitudes comparing formal and informal procedures lead to gains in conceptual knowledge and encouraged these students to “at least attempt to use the formal procedure” (2016, 18). Knowledge of multiple strategies is also key to developing problem solving flexibility, which is important for gains in problem solving (Star & Rittle-Johnson 2008, 566). Flexibility is related to greater abilities to transfer knowledge to new problems along with conceptual knowledge growth (Star & Rittle-Johnson 2008, 566). Alibali et.al (2009) additionally found that direct instruction of certain (but not all types of) new strategies leads to improved problem representation (the representation of the problem in working memory). Accurate problem representation is necessary to correctly solve a problem. Thus, evidence supports that being exposed to new strategies of problem solving through this curriculum could lead to conceptual growth through comparing strategies and developing flexibility, along with improving problem representation.

However, learning new strategies uses cognitive energy and takes time. Hattikudur et. al (2016) found that using creative reasoning led to lower performance during practice but higher performance during a post-test. Star & Rittle-Johnson similarly note that, "when new strategies are learned, it takes time before they are regularly used and benefit performance" (2008, 574). We expected, and saw, that students initially struggled with bar modeling since it was a novel representation and strategy that they needed to spend time learning how to manipulate. While we had expected that this modeling method would be a natural way to represent an equation because it so closely models the symbolic representation of an equation, it was not natural for students and they had to spend extensive time with the tutorials and other instruction. However, we hope that in the long-run, bar modeling will provide students with a better conceptual understanding of algebra and also give them a new tool to use to solve problems in other settings.

**5. Conceptual Change and Common Misconceptions**

One issue brought up in the literature on algebra problem solving is that conceptual change is much more difficult to achieve than conceptual growth (Lucariello et. al 2014). While *conceptual growth* involves assimilating new knowledge to a current conceptual basis, *conceptual change* refers to the process through which conceptual knowledge “must undergo substantial reorganization or replacement” (Lucariello et. al 2014). Thus, misconceptions that students already have coming into algebra can require significant time and emphasis to remedy effectively. Given the limited time a teacher has in their classroom, it is understandable to want to focus on the misconceptions that persist throughout the year of algebra, rather than the misconceptions that dissolve naturally as new concepts are taught (Booth et. al 2014, 10-11). There were three types of the former misconception—those that hinder students’ progress throughout the year and do not dissolve naturally—that we focused on in our workbook: misconceptions about the equals sign, the negative sign, and variables (Busha & Karp 2013).

### 5.1 Equals Sign

In elementary school, students learn the equals sign as a sign meaning “and the answer is?” (Busha & Karp 2013, 620). For example, students are given worksheets full of problems of the form $4+7=$___ or $3*2=$___. Through this practice, the students intuit that the equals sign is an *operational* symbol that means “do something.” In one study, only 6% of students filled in the correct number in the equation $8+4=$___$+5$, while 59% of students reported that the answer is 12 (adding up 8 and 4) and 19% reported that the answer is 17 (adding up 8 and 4 and 5). The stark data highlight the students’ misconception that the equals sign is an operational command, meaning, in this case, “add up the numbers” (Falkner et. al 1999). Additionally, students struggle to interpret a statement such as $3=2+1$, unable to understand an equals sign that comes after a number without an operation (Kerian 1981). This misconception must be addressed in algebra: it only when students understand the equals sign *relationally* as a signal that the two sides are equivalent are they able to manipulate and ultimately solve equations such as $4x+9=2x+15$.

In our bar modeling method, we enforce the relational concept of the equals sign by instructing students to make the bars equal size. In each step, students are told to check that the bars retain equivalency, even as they remove blocks or divide blocks to solve. This would reinforce the concept that the equals sign is relational. Despite this instruction, it was not intuitive for the students to retain equivalent bar lengths. This challenge provides evidence that conceptual change requires *substantial* redirection and reinstruction to allow replacement to take place.

**5.2 Negative Sign**

The second misconception that we address involves the negative sign. According to research from 2013, "negative sign errors persist beyond other types of errors for students enrolled in College Algebra through Calculus II" (Cangelosi et. al 2013). Students struggle to connect the negative sign to a number and prefer to use the negative sign to simply mean subtraction, wherever the subtraction sign may be placed at the end. For example, in a study from 2008, middle school students were interviewed about integer equations. When asked to solve $12-x=7$, many students wrote $x=7-12$ (Vlassis 2008). Here, we see students viewing the negative sign as "subtraction," rather than attaching the negative sign to the $x$.

This misconception gets addressed naturally through the bar modeling method we utilized. In particular, *any* negative sign appears in the bar drawing *attached to a particular number or variable*. For example, given the equation $4x-12=x$, students draw the -12 as a block on top of the 4x block to make the entire bar smaller. The -12 block can only disappear if a +12 block is drawn on top of it, and, to maintain equality, a +12 block must be drawn on the other bar as well. Although the bar modeling method , subtraction was one of the toughest concepts to teach to the students we worked with. Modeling even a simple expression such as *14-5* proved difficult for the students. In particular, students wanted to place the -5 block *beside* the 14 block, visually making the bar bigger, rather than placing the -5 block *on top of* the 14 block to show that the entire bar gets smaller. While this challenge prevented students from reaching conceptual understanding of the negative sign, the students' difficulties lie in a challenge separate from the common misconception. In particular, the difficulty lies in the placement of the blocks, not in the misconception that prevents students from attaching the negative sign to the number or variable.

### 5.3 Variables

Finally, students carry several different misconceptions about variables, including thinking that two variables cannot have the same value, believing the value of the variable is related to its alphabetic position, and viewing a variable as a label (Busha & Karp 2013). Our workbook never dealt with more than one variable in any equation or word problem, which made it difficult to address the first two variable misconceptions listed. It is the last misconception about variables that we focus our attention.

In a study in 1978, middle school students were presented with the following problem: "Cakes cost $c$ pence each and buns cost $b$ pence each. If I buy 4 cakes and 3 buns, what does $4c+3b$ stand for?" (Küchemann 1978). Only 22% of students answered the problem correctly, while 39% of students said that the expression meant "4 cakes and

3 buns" (Küchemann 1978). This study reveals that students often understand the variable as a label (*b* is for buns) rather than as a quantity (*b* is the cost of one bun). But teachers in the United States are inclined to use the mnemonic symbol as variables in word problems in an attempt to facilitate a connection for students between the equation and the real-world problem (McNeil & Weinberg 2010). Unfortunately, a more recent study has found that using nonmnemonic symbols as variables (i.e. $x$ is the cost of one bun) helps students elicit the "letters-as-variables" interpretation, rather than the "letters-as-labels" interpretation (McNeil & Weinberg, 631). The good news is that "findings may not generalize to situations in which students generate their own symbols and write their own algebraic expressions. Indeed, researchers have shown that students' self-generated ways of representing and solving math problems can sometimes lead to very different patterns of performance than those given to students by knowledgeable others" (McNeil & Weinberg, 632).

In our workbook, we made an intentional effort to address this misconception in different ways. The first time that we model how to solve a word problem in the workbook, we step through a scenario involving two women counting the number of basketballs they have all together. We model student thinking in a thought bubble that reads, "What am I asked to find? I am asked to find the number of basketballs they have total. Let's define a variable." Then, we define the variable as "$t$ = total number of basketballs." By this way of modeling how to define the variable, we show the connection between the question ("What am I asked to find? I am asked to find the number of basketballs they have total.") and the variable. Additionally, we chose the variable as $t$, not as $b$, to reinforce that the variable is not a label for "basketballs" but rather represents a number. Finally, and perhaps most importantly, each word problem in our workbook includes space at top specifically for them to choose a variable and to define it. We found that students understood this step with ease, often without explicit instruction: they chose a letter that made sense to them, they defined the variable as a number rather than as a label, they linked the variable definition to what the question was asking. One student even remarked, “Whoa, cool! I can choose any letter!” when asked which letter she would like to use.

Although our short experience using bar modeling with students did not end in perfect redirection from the three common misconceptions described, there are ways that our workbook works to address each misconception. Of course, explicit instruction by the teacher, as well as consistent reinforcement of strategies, should be used in tandem with our workbook to ensure conceptual change is possible.

**6. Conclusions**

While we were unable to systematically evaluate how our curriculum improved math competency and problem solving abilities in students, there is evidence that theoretically this bar modeling technique could encourage gains in conceptual and procedural knowledge along with addressing the major misconceptions students that impede algebra problem solving. With the gained interest in Singapore Math and use of bar modeling in curriculums, research should be done on the cognitive and problem solving benefits of this instruction.

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